EXPONENTIAL LIFE TEST PROCEDURES WHEN THE DISTRIBUTION HAS MONOTONE FAILURE RATE

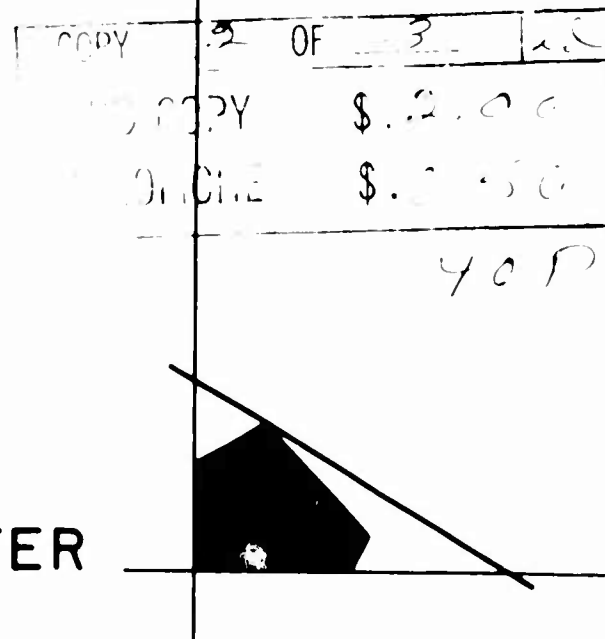by

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ERRATA SHEET for "Exponential Life Test Procedures When the Distribution has Monotone Failure Rate," by R. E. Barlow and Frank Proschan.

Page 11, line 10 and page 12, line 7: Replace by:

$$\hat{R}_1(t) = \begin{cases} (1 - \dfrac{t}{r})^{r-1} & \text{if } t < Z_r \\ 0 & \text{otherwise} \end{cases}$$

Page 12, line 8: Replace by:

$$\geq \begin{cases} 1 - \dfrac{t}{\phi(Z_r^*)} & \text{if } t < Z_r \\ 0 & \text{otherwise}. \end{cases}$$

Page 13, line 6: Replace "Theorem 2.1" by "Corollary 2.2".

Page 13, line 12: Insert "(larger)" after "smaller".

Page 13, 2nd line from bottom:, and page 32, line 5: Replace "$\geq$" by "$\geq (\leq)$".

Page 13, bottom line: Replace "$\leq E$" by "$\leq (\geq) E$".

Page 16, line 3: Insert "$<$" between "T" and "t".

Page 22, line 3: Put subscript "Δ" on "G" and "$\bar{G}$".

Page 22, bottom line:
Replace: "$e^{x \log \frac{q}{\xi_p}}$" by "$e^{\frac{x \log q}{\xi_p}}$".

Page 30, line 14: Replace by "$\bar{H}_i(x)$".

Page 31, line 5: Insert "(DFR)" after "IFR".

EXPONENTIAL LIFE TEST PROCEDURES

WHEN THE DISTRIBUTION HAS MONOTONE FAILURE RATE

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SUMMARY

A number of estimates and tests for mean life and other parameters derived under the exponential distribution assumption are studied under the alternative condition that the distributio. has increasing (decreasing) failure rate. The estimates considered are, for the most part, based on censored and truncated samples. It is shown that these estimates generally favor the producer (consumer) in the IFR (DFR) case. Properties of order statistics and their spacings from distributions with increasing (decreasing) failure rate are presented.

1.   Introduction. In a fundamental paper in the literature of life testing Epstein and Sobel (1953) introduce life test procedures based on the exponential distribution. These procedures have been codified in a Department of Defense handbook (1960) and are now widely employed. Zelen and Dannemiller (1961) show by sampling from Weibull distribution alternatives that these procedures are not robust in testing for mean life. However, as Antelman and Savage (unpub.) have pointed out they may be robust in testing for cert..n percentiles. For certain loss functions based on percentiles, these procedures seem to be robust. Since statistical procedures based on the exponential distribution have a great deal of intuitive appeal and computational simplicity we investigate their properties relative to alternative distributions having increasing failure rate (IFR) or decreasing failure rate (DFR).

This paper essentially confirms, theoretically and more generally, the sampling results of the Zelen-Dannemiller paper for statistics derived under the exponential assumption. Using Weibull distribution alternatives (with parameter values which insure that the distribution has increasing failure rate) Zelen-Dannemiller show that the use of these statistics may result in substantially increasing the probability of accepting items having poor mean lives. We show that these estimates for the mean are positively (negatively) biased when the distribution is IFR (DFR). Also we obtain bounds on the expected values of the exponential estimates for the distribution function and bounds on the expected values of the order statistics.

In the last section various properties of IFR (DFR) order statistics are presented.

Preliminaries. Let X denote a random variable with right continuous distribution F such that $F(0^-) = 0$. If F has density f then $r(t) = \frac{f(t)}{[1 - F(t)]}$ is known as the failure rate. Note that $r(t) = -\frac{d}{dt} \log[1 - F(t)]$ when a density exists. For this reason, we say that F is IFR (DFR) for increasing (decreasing) failure rate if $\log[1 - F(t)]$ is concave where finite (convex on $[0, \infty)$). Note that any IFR (DFR) distribution with specified mean can be expressed as the limit of continuous IFR (DFR) distributions with the same mean. Hence for many of our results it is sufficient to confine attention to continuous IFR (DFR) distributions.

We often use the well known fact that if F is continuous, then $Y = -\theta \log \bar{F}(X)$ is exponentially distributed with mean θ where $\bar{F}(x) = 1 - F(x)$. Repeatedly we use the fact that if F is IFR with mean θ then there exists $x_0 \geq \theta$ such that

$$y = -\theta \log \bar{F}(x) \begin{cases} < x & \text{for } x < x_0 \\ \geq x & \text{for } x \geq x_0 \end{cases} .$$

This is evident from log concavity and the bounds on IFR distributions given in Barlow and Marshall (1964). The inequalities are reversed when F is DFR.

Unless otherwise indicated we denote ordered observations from a random sample of size n based on a random variable X

by $X_1 \leq \ldots \leq X_n$. We define $X_0 = 0$.

2. Estimates based on censored samples.

Assume n items are put on life test and let $X_1 \leq X_2 \leq \ldots \ldots \leq X_n$ denote the ordered observations. If F has density f such that

$$f(x; \theta) = \begin{cases} \dfrac{1}{\theta} e^{-\frac{x}{\theta}} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

then

$$(2.1) \qquad \hat{\theta}_{r, n} = \frac{\sum_1^r X_i + (n - r)X_r}{r} = \frac{\sum_1^r (n - i + 1)(X_i - X_{i-1})}{r}$$

$(1 \leq r \leq n)$ is the maximum likelihood and minimum variance unbiased estimate for θ based on the first r order statistics (Epstein and Sobel, 1953).

The normalized spacings $\bar{D}_i = (n - i + 1)(X_i - X_{i-1})$ which enter into the computation of $\hat{\theta}_{r, n}$ have a natural intuitive appeal. They have also been used as the basis for a statistic to test for IFR (Proschan and Pyke, in preparation). We shall derive and use several properties of these spacings when F is IFR (DFR) . Since the normalized spacings are independent and identically distributed in the exponential case (Epstein and Sobel, 1953), Theorem 2.1 below is quite intuitive.

A random variable X is said to be stochastically smaller than a random variable Y if and only if $P[X \geq x] \leq P[Y \geq x]$ for all x.

Theorem 2.1. If F is IFR (DFR) the normalized spacings $(n - i + 1)(X_i - X_{i-1})$ are stochastically decreasing (increasing) in i.

Proof. Assume F is IFR and let $\bar{F}(x) = 1 - F(x)$. Note that

$$P[nX_1 \geq x] = [\bar{F}(\frac{x}{n})]^n \geq [\bar{F}(\frac{x}{n-1})]^{n-1}$$

since $[\bar{F}(t)]^{\frac{1}{t}}$ is decreasing in t. Let

$$F_u(x) = \frac{F(u + x) - F(u)}{\bar{F}(u)}$$

and note $F_u(x) \geq F(x)$. Given that $X_1 = u$ is observed, $X_2 - X_1$ is distributed as the first order statistic from a sample of size n - 1 each with distribution $F_u(x)$. Hence

$$P[(n - 1)(X_2 - X_1) \geq x \mid X_1 = u] = [\bar{F}_u(\frac{x}{n-1})]^{n-1} .$$

Conditioning on X_1 we have

$$P[nX_1 \geq x] = [\bar{F}(\frac{x}{n})]^n \geq [\bar{F}_u(\frac{x}{n-1})]^{n-1}$$

$$= P[(n - 1)(X_2 - X_1) \geq x \mid X_1 = u]$$

for all $u \geq 0$. Unconditioning

$$P[nX_1 \geq x] \geq \int_0^\infty [\bar{F}_u(\frac{x}{n-1})]^{n-1} dG(u)$$

$$= P[(n-1)(X_2 - X_1) \geq x]$$

where $G(u) = 1 - [\bar{F}(u)]^n$ is the distribution of X_1. Hence we have shown that nX_1 is stochastically larger than $(n - 1)(X_2 - X_1)$. In a similar manner we can show that $(n - i + 1)(X_i - X_{i-1})$ is stochastically larger than $(n - i)(X_{i+1} - X_i)$ for $i = 2, 3, \ldots, n$.

All inequalities are reversed for DFR distributions. $\|$

As an immediate consequence of Theorem 2.1 we have that

$$E\left\{ \left[(n - i + 1)(X_i - X_{i-1}) \right]^{\alpha} \right\}$$

is decreasing (increasing) in i for $\alpha \geq 0$ when F is IFR (DFR) . Using this fact we can show that $\hat{\theta}_{r, n}$ is positively biased when F is IFR.

Corollary 2.2. If F is IFR with mean θ , then

$$\theta \leq E[\hat{\theta}_{r, n}] \leq n\frac{\theta}{r} \qquad \text{for} \quad r = 1, 2, \ldots, n.$$

All inequalities are sharp.

Proof. From Barlow and Proschan (1964a, p.33) we know that $E[\hat{\theta}_{1, n}] \geq \theta$. Also

$$h(r) = \sum_{i=1}^{r} \left\{ E[(n - i + 1)(X_i - X_{i-1})] - \theta \right\}$$

exhibits at most one sign change as a function of r since

$$E[(n - i + 1)(X_i - X_{i-1})]$$

is decreasing in i by Theorem 2.1. But $h(1) \geq 0$ and $h(n) = 0$, which implies $h(r) \geq 0$ for $r = 1, 2, \ldots, n$. Hence

$$E[\hat{\theta}_{r, n}] \geq \theta \ .$$

Clearly the bound is attained by the exponential distribution so that it is sharp.

To show the upper bound we note

$$\sum_{i=1}^{r} X_i + (n - r) X_r \leq \sum_{i=1}^{n} X_i$$

for every sample realization. Hence

$$E[r\hat{\theta}_{r, n}] \leq n \theta$$

or $\qquad\qquad E[\hat{\theta}_{r, n}] \leq n\frac{\theta}{r} \ .$

Since equality is attained with distributions degenerate at θ (which is the limit of IFR distributions) the bound is sharp. $\|$

Corollary 2.3. If F is DFR with mean θ, then

$$0 \leq E[\hat{\theta}_{r,n}] \leq \theta \qquad \text{for} \quad 1 \leq r < n .$$

All inequalities are sharp.

Proof. The upper bound follows from Theorem 2.1 and the method of proof in Corollary 2.2. To show that the lower bound is sharp, let

$$\overline{F}(x) = \begin{cases} 0 & x < 0 \\ \epsilon \, e^{-\frac{\epsilon x}{\theta}} & x \geq 0 \end{cases}$$

where $\epsilon > 0$ is arbitrary. Then F is DFR with mean θ and

$$P[X_i \geq x] = \sum_{j=0}^{i-1} \binom{n}{j} [F(x)]^j [\overline{F}(x)]^{n-j}$$

$$\leq \sum_{j=0}^{i-1} \binom{n}{j} \epsilon^{n-j} e^{-\frac{\epsilon x(n-j)}{\theta}}$$

Hence

$$E[X_i] = \int_0^\infty P[X_i \geq x] \, dx \leq \sum_{j=0}^{i-1} \binom{n}{j} \frac{\epsilon^{n-j-1} \theta}{(n-j)} < 2^n \epsilon \, \theta$$

when $0 \leq \epsilon < 1$. Since ϵ is arbitrary we see that

$$E[X_i] \geq 0 \qquad\qquad (1 \leq i < n)$$

is sharp. $\|$

For convenience we now denote the ith order statistic from a sample of size n by $X_{i,n}$.

Theorem 2.4. If F is IFR (DFR) $(n - i + 1)(X_{i,n} - X_{i-1,n})$ is stochastically increasing (decreasing) in n for fixed i. Hence

$$E[\hat{\theta}_{r,n}] \underset{(\geq)}{\leq} E[\hat{\theta}_{r,n+1}] . \qquad\qquad (1 \leq r \leq n)$$

Proof. Assume F is IFR . Let $G_{i,n}(x) = P[X_{i,n} \leq x]$ and note that $G_{i,n}(x) \leq G_{i,n+1}(x)$ for samples from any distribution. Now

$$P[(n-i)(X_{i+1,n} - X_{i,n}) \geq x] = \int_0^\infty [F_u(\frac{x}{n-i})]^{n-i} dG_{i,n}(u)$$

$$\leq \int_0^\infty [F_u(\frac{x}{n+1-i})]^{n+1-i} dG_{i,n}(u)$$

$$\leq \int_0^\infty [F_u(\frac{x}{n+1-i})]^{n+1-i} dG_{i,n+1}(u)$$

$$= P[(n+1-i)(X_{i+1,n+1} - X_{i,n+1}) \geq x] ;$$

the first inequality holds since $[F(t)]^{\frac{1}{t}}$ is decreasing in t when F is IFR .

All inequalities are reversed when F is DFR. ||

Thus when F is IFR , the estimate, $\hat{\theta}_{r,n}$, of mean life based on a sample censored on the right becomes worse with increasing n when r $(1 \leq r \leq n)$ remains fixed.

Acceptance Sampling. Statistical methods for testing hypotheses about the mean of an exponential distribution depend on the statistic, $\hat{\theta}_{r,n}$, in the case of censored samples (Epstein, 1960a). For testing the hypothesis $H_0: \theta = \theta_0$ against the alternative $H_1:$ $\theta = \theta_1 < \theta_0$ subject to $P[\text{reject } \theta = \theta_0 \mid \theta_0 \text{ true}] = \alpha$, the rejection region is of the form

$$\hat{\theta}_{r,n} \leq \frac{\theta_0 \chi^2_{1-\alpha}(2r)}{2r} .$$

If $\chi^2_{1-\alpha}(2r) < 2r$, then we shall prove, using Lemma 2.5, that

$$(2.2) \qquad P\left[\ \hat{\theta}_{r,n} \le \frac{\theta_0 \chi^2_{1-\alpha}(2r)}{2r} \qquad F \ IFR, \ \theta = \theta_0 \right]$$

$$\le \ P\left[\ \hat{\theta}_{r,n} \le \frac{\theta_0 \chi^2_{1-\alpha}(2r)}{2r} \ \bigg| \ F(x) = 1 - e^{-\frac{x}{\theta_0}} \right] \ = \ \alpha$$

so that the exponential test provides a size α test when the failure distribution is IFR . To see (2.2) we need the following easily verified result which we present without proof.

Lemma 2.5. If ϕ is concave $\phi(0) = 0$, and $a_i \ge 1$, $x_i \ge 0 \ (i = 1, 2, \ldots, n)$, then

$$\phi\left[\ \sum_{i=1}^{n} a_i x_i \right] \le \sum_{i=1}^{n} a_i \ \phi(x_i) \ .$$

Let $\phi^{-1}(y) = -\theta \log \overline{F}(y)$ so that $X_i = \phi(Y_i)$ where Y_i is the ith order statistic of an exponentially distributed random variable with mean θ . Then

$$r\hat{\theta}_{r,n} = \sum_{i=1}^{r} \phi(Y_i) + (n - r) \phi(Y_r)$$

$$\ge \phi\left[\ \sum_{i=1}^{r} Y_i + (n - r)Y_r \right]$$

by Lemma 2.5. Using the bounds on IFR distributions (Barlow and Marshall, 1964) and letting $Z_r^* = \sum_{i=1}^{r} Y_i + (n - r)Y_r$, we have

$$P_\theta \left[\phi(Z_r^*) \le c\right] \le P_\theta \left[Z^* \le c\right]$$

when $c < \theta$. We obtain (2.2) by letting $\theta = \theta_0$ and

$$c = \frac{\theta_0 \times \chi^2_{1-\alpha}(2r)}{2} \quad .$$

<u>Sampling with Replacement</u>. Suppose now that failed items are re-placed at failure. In this case the bias of the usual estimate for θ is even greater than in the non-replacement case.

Let X^*_i denote the time of the ith failure when failed items are replaced. The maximum likelihood estimate for θ based on the exponential assumption is, in this case

$$\hat{\theta}^*_{r,n} = \frac{1}{r} \left[nX^*_1 + n(X^*_2 - X^*_1) + \ldots + n(X^*_r - X^*_{r-1}) \right]$$

$$= \frac{nX^*_r}{r}$$

(Epstein, 1960b).

<u>Theorem 2.5.</u> If F is IFR with mean θ, then

$$\theta \le E[\hat{\theta}_{r,n}] \le E[\hat{\theta}^*_{r,n}] \le \frac{n}{r} E[X_r] \quad 1 \le r \le n \quad .$$

<u>Proof.</u> Clearly $X^*_r \le X_r$ for any distribution F so that the upper bound is obvious. To show the remaining inequality we intro-duce the following fictitious replacement policy:

Policy A: Replace a failed item with a good item of the same "age".
Let X^{**}_i denote the time of the ith failure under this policy and

$$\hat{\theta}^{**}_{r,n} = \frac{1}{r} \sum_{i=1}^{r} (n - i + 1)(X^{**}_i - X^{**}_{i-1}) \quad .$$

It is clear that $E[\hat{\theta}^{**}_{r,n}] \le E[\hat{\theta}^*_{r,n}]$

since under the IFR assumption the conditional mean life of an aged item is less than the mean life of a new item.

We need only show $E[\hat{\theta}_{r,n}] \leq E[\hat{\theta}_{r,n}^{**}]$. Let

$$F_u(x) = \frac{F(x + u) - F(u)}{\bar{F}(u)} \qquad \text{and} \qquad G(u) = P[X_1 \leq u] . \text{ Then}$$

$$P[X_2 - X_1 \geq \frac{x}{n-1} \mid X_1 = u] = [F_u(\frac{x}{n-1})]^{n-1} \quad \text{and}$$

$$E[(n-1)(X_2 - X_1)] = \int_0^\infty \int_0^\infty [F_u(\frac{x}{n-1})]^{n-1} \, dx \, dG(u) . \text{ Similarly}$$

under Policy ..

$$P[X_2^{**} - X_1^{**} \geq \frac{x}{n} \mid X_1 = u] = [F_u(\frac{x}{n})]^n$$

and

$$E[n(X_2^{**} - X_1^{**})] = \int_0^\infty \int_0^\infty [F_u(\frac{x}{n})]^n \, dx \, dG(u) .$$

Since $\quad [F_u(\frac{x}{n})]^n \geq [F_u(\frac{x}{n-1})]^{n-1} \quad$ when $\quad F \quad$ is $\quad$ IFR $\quad$ we

have

$$E[(n-1)(X_2 - X_1)] \leq E[n(X_2^{**} - X_1^{**})] .$$

To show

$$E[(n - i + 1)(X_i - X_{i-1})] \leq E[(n - i + 1)(X_i^{**} - X_{i-1}^{**})]$$

for $\quad 2 < i \leq n \quad$ we proceed as above except that the definition of $\quad G$ is different for the two policies. For example, for $\quad i = 3 \quad$ let

$$G(u_2 \mid u_1) = P[X_2 \leq u_2 \mid X_1 = u_1]$$

$$= 1 - [F_{u_1}(u_2 - u_1)]^{n-1}$$

and

$$G^*(u_2 \mid u_1) = P[X_2^{**} \leq u_2 \mid X_1^{**} = u_1]$$

$$= 1 - [F_{u_1}(u_2 - u_1)]^n .$$

Since $\quad G(u_2 \mid u_1) \leq G^*(u_2 \mid u_1) \quad$ we have

$$E[(n-2)(X_3 - X_2)]$$

$$= \int_0^\infty \int_0^\infty [\bar{F}_{u_2}(\frac{x}{n-2})]^{n-2} dx \, dG(u_2 \mid u_1) \, dG(u_1)$$

$$\leq \int_0^\infty \int_0^\infty [\bar{F}_{u_2}(\frac{x}{n-2})]^{n-2} dx \, dG^*(u_2 \mid u_1) \, dG(u_1)$$

$$\leq \int_0^\infty \int_0^\infty [\bar{F}_{u_2}(\frac{x}{n})]^n dx \, dG^*(u_2 \mid u_1) \, dG(u_1)$$

$$= E[n(X_3^* - X_2^*)] \quad . \text{ A similar conditioning argument}$$

works for $3 < i \leq n$. ||

Bounds on estimates for the reliability function. The minimum
variance unbiased estimate for $R(t) = \bar{F}(t)$ (t is fixed) under the
exponential assumption is

$$\hat{R}_1(t) = \max[0, (1 - \frac{t}{Z_r})^{r-1}]$$

where $Z_r = \sum_{i=1}^r X_i + (n-r)X_r$. For a discussion of such minimum

variance unbiased estimates see Tate (1959). For convenience,
assume that $\theta = 1$. Then, under the exponential assumption, Z_r
has density

$$g_r(y) = \frac{y^{r-1} e^{-y}}{(r-1)!} \quad .$$

Theorem 2.6. If F is IFR with mean $\theta = 1$ and $t < \theta = 1$,
then

$$E[\hat{R}_1(t)] \geq \int_t^1 [1 - \frac{t}{y}]^{r-1} g_r(y) \, dy + \int_1^\infty (1-t)^{r-1} g_r(y) dy \quad .$$

Proof. Without loss of generality we may assume F continuous
(see preliminaries). Let $\phi^{-1}(y) = -\log \bar{F}(y)$. Then ϕ is con-
cave, increasing and $\phi(0) = 0$. If Y_i is the ith order statistic

from an exponentially distributed random variable with $\theta = 1$,

then $X_i = \phi(Y_i)$ is the ith order statistic from an IFR random

variable with distribution F and mean $\theta = 1$. Furthermore

$$Z_r = \sum_{i=1}^{r} \phi(Y_i) + (n - r) \phi(Y_r)$$

$$\geq \phi\left[\sum_{i=1}^{r} Y_i + (n - r)Y_r\right] = \phi[Z_r^{*}]$$

by the previous lemma. Therefore

$$\hat{R}_1(t) = \max\left[0, (1 - \frac{t}{Z_r})^{r-1}\right]$$

$$\geq \max\left[0, (1 - \frac{t}{\phi(Z_r^{*})})^{r-1}\right] .$$

Since

$$\phi(y) \geq \begin{cases} y & y < 1 \\ 1 & y \geq 1 \end{cases}$$

we have

$$E[\hat{R}_1(t)] \geq \int_t^1 [1 - \frac{t}{y}]^{r-1} g_r(y)\, dy + \int_1^\infty (1 - t)^{r-1} g_r(y)\, dy .$$

The maximum likelihood estimate for $R(t)$ under the exponential assumption is

$$\hat{R}_2(t) = e^{-\frac{t}{\hat{\theta}_{r,n}}}$$

where $\hat{\theta}_{r,n}$ was defined in (2.1). Pugh (1963) has shown that under the exponential assumption $\hat{R}_2(t)$ is negatively biased when the true reliability $R(t) > \frac{1}{e} = .368$. Assuming F is IFR we can obtain a lower bound on $E[\hat{R}_2(t)]$.

Theorem 2.7. If F is IFR with mean $\theta = 1$, then

$$E[\hat{R}_2(t)] \geq \int_0^1 e^{-\frac{tr}{(y\theta)}} g_r(y)\, dy + e^{-\frac{tr}{\theta}} \int_1^\infty g_r(v)\, dv .$$

The proof parallels that of Theorem 2.6.

Estimates and confidence bounds on percentiles. If F is IFR with mean θ and p^{th} percentile ξ_p, then

$$[-\log(1 - p)]\theta \leq \xi_p \leq \frac{[-\log(1 - p)]}{p}\theta .$$

(See Barlow and Marshall (1964)). Hence by Theorem 2.1

$$E\left\{\hat{\theta}_{r,n} \frac{[-\log(1 - F)]}{p}\right\} \geq \xi_p \qquad \text{for} \quad = 1, 2, \ldots, n$$

while

$$E\left\{\hat{\theta}_{n,n} [-\log(1 - p)]\right\} \leq \xi_p$$

and one might be tempted to use these estimates to bracket ξ_p.

Intuitively, we want a confidence interval to have small expected width when it covers the true percentile. The usual distribution-free confidence intervals based on order statistics have smaller conditional expected width under the IFR (DFR) assumption than under the exponential assumption, given that the interval contains the true percentile. To see this let $Y = -\frac{\xi_p \log F(X)}{-\log(1 - p)}$ and note that Y is exponentially distributed with p^{th} percentile ξ_p when F is continuous. Suppose that $X_i \leq \xi_p \leq X_j$. Then clearly $\frac{Y_j - Y_i}{X_j - X_i} \geq 1$, which implies

$$E[X_j - X_i \mid X_i \leq \xi_p \leq X_j] \leq E[Y_j - Y_i \mid Y_i \leq \xi_p \leq Y_j] .$$

3. **Estimates based on truncated samples.** If n items are placed on life test and if sampling is terminated at time T, the associated sample is called a truncated sample. Let $X_1 \leq X_2 \leq \ldots \leq X_n$ denote an ordered sample from a distribution F and let

$$V(T) = \sum_{i=1}^{r} X_i + (n - r)T$$

where r is a random variable and denotes the number of X's less than T. Then $V(T)$ is the total life observed up to time T. This statistic occurs, for example, in sequential life tests for the exponential case (Epstein and Sobel, 1955). It is not surprising that this statistic also has greater expected value under the IFR assumption.

For convenience, let $G(x) = 1 - e^{-\frac{x}{\theta}}$.

Theorem 3.1. If F is IFR (DFR) with mean θ, then

$$E_F[V(T)] \underset{(\leq)}{\geq} E_G[V(T)]$$

Proof. Assume F IFR and let $X_1 \leq X_2 \leq \ldots \leq X_n$ denote an ordered sample from F. Without loss of generality we may assume F continuous. Let $y = -\theta \log \bar{F}(x)$. We know there exists $x_0 > \theta$ such that $x \geq -\theta \log \bar{F}(x)$ for $x \leq x_0$ and $x \leq -\theta \log \bar{F}(x)$ for $x > x_0$ (Barlow and Marshall, 1964).

Let $Y_i = -\theta \log \bar{F}(X_i)$. If $T \leq x_0$, then

$$\sum_{i=1}^{r} X_i + (n - r)T \geq \sum_{i=1}^{r} Y_i + (n - r)T \geq \sum_{i=1}^{s} Y_i + (n - s)T$$

where r (s) denotes the number of X's $(Y$'s$)$ less than T.

Hence for $T \leq x_0$

$$E_F[V(T)] \geq E_G[V(T)] \quad .$$

Let

$$Y_i^* = \begin{cases} Y_i & \text{if } Y_i \leq T \\ T & \text{otherwise.} \end{cases}$$

For $T > x_0$,

$$\sum_{i=1}^{r} X_i + (n - r)T - \sum_{i=1}^{s} Y_i - (n - s)T$$

$$= \sum_{i=1}^{r} X_i + (n - r)T - \sum_{i=1}^{r} Y_i^* - (n - r)T$$

$$\geq \sum_{i=1}^{r} (X_i - Y_i) + \sum_{i=r+1}^{n} (X_i - Y_i)$$

since $X_i \leq Y_i$ for $i > r$. Hence

$$E_F[V(T)] - E_G[V(T)] \geq E\left[\sum_{i=1}^{n} X_i\right] - E\left[\sum_{i=1}^{n} Y_i\right] = 0$$

for $T > x_0$.

A similar argument holds for the DFR case.

Consider the estimate

$$\hat{\theta}(T) = \frac{V(T)}{r} = \begin{cases} \dfrac{\sum\limits_{i=1}^{r} X_i + (n - r)T}{r} & \text{if } r > 0 \\ nT & \text{if } r = 0 \end{cases}$$

When F is the exponential distribution, $\hat{\theta}(T)$ is the maximum likelihood estimate of θ . In this case

$$E[\hat{\theta}(T)] = \theta - \frac{\text{cov}(r, \hat{\theta}(T))}{1 - \exp(-\frac{T}{\theta})} > \theta$$

since r and $\hat{\theta}(T)$ are negatively correlated (Bartholomew, 1957). In the IFR case, this statistic exhibits even greater bias for $T < \theta$. As before let $G(x) = 1 - e^{-\frac{x}{\theta}}$.

Theorem 3.2. If F is IFR with mean θ, but not degenerate then

$$E_F[\hat{\theta}(T) \mid r \geq 1] \geq E_G[\hat{\theta}(T) \mid r \geq 1] \quad \text{for} \quad T < \theta.$$

Proof. Assume F is continuous and let $Y_i = -\theta \log \bar{F}(X_i)$ as before and let

$$a_i = \begin{cases} 1 & \text{if } X_i \leq T \\ 0 & \text{otherwise} \end{cases}$$

$$b_i = \begin{cases} 1 & \text{if } Y_i \leq T \\ 0 & \text{otherwise} \end{cases}.$$

We can write

$$\hat{\theta}(T) = \frac{\sum_{i=1}^{n} \left[T - a_i (T - X_i) \right]}{\sum_{i=1}^{n} a_i} \quad \text{if} \quad \sum_{i=1}^{n} a_i \geq 1.$$

Assume $T < \theta$. As in the previous proof $X_i < T$ implies $Y_i < T$ and hence $a_i \leq b_i$. If $a_i = 1$, then $b_i = 1$ and $T - X_i < T - Y_i$. Hence if $\sum_{i=1}^{n} a_i \geq 1$,

$$\frac{\sum_{i=1}^{n} \left[T - a_i(T - X_i) \right]}{\sum_{i=1}^{n} a_i} > \frac{\sum_{i=1}^{n} \left[T - b_i(T - Y_i) \right]}{\sum_{i=1}^{n} b_i}$$

and

$$E_F[\hat{\theta}(T) \mid r \geq 1] \geq E_G[\hat{\theta}(T) \mid r \geq 1] \quad \text{for} \quad T \leq \theta. \;\|$$

Inverse Binomial Sampling.

Nadler (1960) has considered the following type of sampling.
An item having life distribution F with mean θ is put on test until
it fails or time t has elapsed; at this time the item is replaced by a
fresh item. This is repeated sequentially until r actual failures
are observed. The number N . of items that have to be tested until
the r actual failures are obtained is a random variable. Nadler
(1960) showed that when $F(x) = 1 - e^{-\frac{x}{\theta}}$, an unbiased estimate of
θ is

$$\hat{\theta}_r(t) = \frac{1}{r} \left\{ \sum_{i=1}^{r} Y_i + (N-r)t \right\}$$

where the $Y_1, \ldots, Y_r$ are the r life lengths not exceeding t .

We show next that when F is IFR (DFR) with mean θ , then
$\hat{\theta}_r(t)$ is biased high (low).

Theorem 3.3. If F is IFR (DFR) with mean θ , then $E\hat{\theta}_r(t) \geq (\leq) \theta$.

Proof. Let F be IFR . Let Z_i denote test time elapsed between
the $i - 1^{st}$ failure time and the ith failure time, $i = 1, 2, \ldots, r$,
where the 0th failure time is defined to be 0 . Then

$$\hat{\theta}_r(t) = \frac{1}{r} \sum_{i=1}^{r} Z_i \ .$$

Next consider an alternate testing procedure differing in that
replacement occurs only upon failure. Let Z_i' = test time elapsed
between the $i - 1^{st}$ failure and the ith failure under the alternate
testing procedure. Now since F is IFR , Z_i is stochastically
larger than Z_i' . It follows that

$$E\hat{\theta}_r(t) = \frac{1}{r} \sum_{i=1}^{r} E Z_i \geq \frac{1}{r} \sum_{i=1}^{r} E Z_i' = \theta$$

The inequality is reversed when F is DFR.

<u>Sampling with replacement</u>. In this case

$$\hat{\theta}(T) = \frac{nT}{r} = \frac{nT}{\sum_{i=1}^{n} N_i(T)}$$

where $N_i(T)$ denotes the number of replacements in the ith item position and $r = \sum_{i=1}^{n} N_i(T)$ denotes the total number of replacements in $[O, T]$. Of course $E\left[\dfrac{1}{\sum_{i=1}^{n} N_i(T)}\right]$ is unbounded.

However, we know that $E[N_i(T)] \leq \dfrac{T}{\theta}$ for all $T > 0$ (Barlow and Proschan, 1964b). Hence this again indicates that $\hat{\theta}(T)$ will tend to be larger in the IFR case than in the exponential case.

<u>4. Bounds on time to r^{th} failure.</u> Under the exponential assumption, the distribution of the statistic $\hat{\theta}_{r,n}$ depends only on r and not on n. The choice of n in this case is usually determined by the ratio

$$\frac{E(X_{r,n})}{E(X_{r,r})}$$

which is an indirect measure of the expected saving in time due to putting more than r items on test but terminating at the rth failure (Epstein, 1960a). We always have

$$\frac{E(X_{r,n})}{E(X_{r,r})} \leq 1 .$$

Since the bound is attained by the degenerate distribution (which is the limit of IFR distributions), this is not a useful measure if we assume only IFR . However, we can obtain non-trivial bounds on $E(X_{r,n})$.

Assume F is IFR, with mean 1, and continuous. We may write $Y_i = -\log \bar{F}(X_i)$ where Y_i is the ith order statistic in a sample of n from distribution $G(x) = 1 - e^{-x}$, and is a convex function of X_i, where X_i is the ith order statistic in a sample of n from F . By Jensen's inequality

$$E(Y_i) \geq -\log \bar{F}[E(X_i)],$$

so that

$$\bar{F}[E(X_i)] \geq e^{-E(Y_i)} .$$

If $b(x)$ is a sharp upper bound on $\bar{F}(x)$, then b is decreasing to 0 and

$$b[E(X_i)] \geq e^{-E(Y_i)} .$$

Hence choosing x_0 such that

$$b(x_0) = e^{-E(Y_i)} ,$$

where of course

$$E(Y_i) = \sum_{j=1}^{i} \frac{1}{(n-j+1)} ,$$

we have

$$E(X_i) \leq x_0 .$$

Using tabled upper bounds on $\bar{F}$ given one or two moments of F (Barlow and Marshall, 1963) we can obtain upper bounds on $E(X_i)$. When F is DFR we can, in a similar manner, obtain lower bounds on $E(X_i)$ using lower bounds on $\bar{F}$.

If we specify the first moment of F, explicit upper bounds can be given on $E(X_i)$ when F is IFR, as shown in

<u>Theorem 4.1.</u> If $X_1 \leq X_2 \leq \cdots \leq X_n$ are order statistics from an IFR random variable with mean θ and $Y_1 \leq Y_2 \leq \cdots \leq Y_n$ are the order statistics from $G(x) = 1 - e^{-x}$, then

(a) $\quad \theta E(Y_1) \leq E(X_1) \leq \theta$

(b) $\qquad E(X_i) \leq \dfrac{\theta E(Y_i)}{1 - e^{-E(Y_i)}} \qquad 1 < i < n$

(c) $\qquad \theta \leq E(X_n) \leq \theta E(Y_n)$

(a) and (c) are sharp and (b) is non-trivial though not sharp.

<u>Proof.</u> (a) and (c) are shown in Barlow and Proschan (1964a), Chapter 2. Hence we need only prove (b) and we may assume $\theta = 1$.

First let us verify that (b) is non-trivial. Note that by (a), $EX_{n-1} \leq EX_n \leq EY_n$, so that EY_n is a trivial upper bound for EX_{n-1}. Therefore, a non-trivial upper bound for EX_{n-1} must be less than EY_n; i.e., we must show that $\dfrac{EY_{n-1}}{1 - e^{-EY_{n-1}}} < EY_n$.

But for $z > 0$, $\dfrac{z}{1 - e^{-z}} < z + 1$; thus letting $z = EY_{n-1} = \dfrac{1}{n} + \dfrac{1}{n-1} + \cdots + \dfrac{1}{2}$, we conclude

$$\frac{EY_{n-1}}{1 - e^{-EY_{n-1}}} < \frac{1}{n} + \frac{1}{n-1} + \cdots + \frac{1}{2} + 1 = EY_n \quad .$$

To show (b) use the bound

$$\bar{F}(x) \leq b(x) = \begin{cases} 1 & x \leq \theta \\ e^{-wx} & x > \theta \end{cases}$$

where w depends on x and satisfies $\int_0^x e^{-wu}\, du = 0$.

(Barlow and Marshall, 1964). ‖

Sharp bounds will be derived in a future publication. However these are not as convenient as the bounds of Theorem 4.1.

Bounds on expected values of order statistics can also be given in terms of the pth percentile.

Theorem 4.2. Let $X_1 \leq X_2 \leq \dots \leq X_n$ denote the order statistics from F , IFR with p^{th} percentile ξ_p . Then

(4.1)
$$E(X_j) \leq \max\left\{ \xi_p,\ \frac{\xi_p}{-\log q}\left(\frac{1}{n} + \dots + \frac{1}{(n-j+1)}\right) \right\}$$

and

(4.2)
$$E(X_j) \geq \sum_{i=0}^{j-1} \binom{n}{i} \int_0^{\xi_p} \left(1 - e^{\frac{x \log q}{\xi_p}}\right)^i \left(e^{\frac{x \log q}{\xi_p}}\right)^{n-i} dx$$

where $q = 1 - p$. All inequalities are sharp.

Proof. To show (4.1), let

$$\bar{G}_\Delta(x) = \begin{cases} 1 & 0 \leq x \leq \Delta \\ q \exp\left[\left(\frac{x-\xi_p}{\xi_p-\Delta}\right)\log q\right] & x > \Delta \end{cases}.$$

Note that $\bar{G}_\Delta(\Delta) = 1$ and $\bar{G}_\Delta(\xi_p) = 1 - p = q$. Since $\log \bar{F}(x)$ is concave, there exists at least one value of $\Delta \geq 0$ such that $\bar{G}_\Delta(x) \geq \bar{F}(x)$ for all $x \geq 0$. Thus $E(X_j) \leq \sup_{0 \leq \Delta \leq \xi_p} E(Y_j)$ where

Y_j is the jth order statistic from G_Δ . Now

$$E(Y_j) = \Delta + \int_\Delta^\infty \sum_{i=0}^{j-1} \binom{n}{i} [G_\Delta(x)]^i [\bar{G}_\Delta(x)]^{n-i} dx$$

$$= \Delta + \int_\Delta^\infty \frac{\Gamma(n+1)}{\Gamma(j)\Gamma(n+1-j)} \int_{G_\Delta(x)}^1 t^{j-1}(1-t)^{n-j} dt\, dx$$

To find the maximizing Δ , consider

$$\frac{\partial}{\partial\Delta} E(Y_j) = 1 - \frac{\Gamma(n+1)}{\Gamma(j)\Gamma(n+1-j)} \int_{G_\Delta(\Delta)}^{1} t^{j-1}(1-t)^{n-j}\, dt$$

$$+ \frac{\Gamma(n+1)}{\Gamma(j)\Gamma(n+1-j)} \int_\Delta^\infty [G(x)]^{j-1}[\bar{G}(x)]^{n-j}$$

$$\cdot q \exp[\frac{x-\xi_p}{\xi_p - \Delta}\log q]\ \log q\, \frac{x-\xi_p}{(\xi_p - \Delta)^2}\, dx\ .$$

Since $G_\Delta(\Delta) = 0$, $-(\xi_p - \Delta)\frac{\partial E(Y_j)}{\partial\Delta}$ reduces to

$$\frac{\Gamma(n+1)}{\Gamma(j)\Gamma(n+1-j)} \int_\Delta^\infty [G_\Delta(x)]^{j-1}[\bar{G}_\Delta(x)]^{n-j} g_\Delta(x)(x-\xi_p)\, dx = E(Y_j)-\xi_p$$

where g_Δ is the density of G_Δ .

Hence

$$- (\xi_p - \Delta)\frac{\partial}{\partial\Delta} E(Y_j) = \Delta - \frac{\xi_p - \Delta}{\log q}\ (\frac{1}{n}+\ldots+\frac{1}{n-j+1}) - \xi_p$$

$$= -(\xi_p - \Delta)[1 + \frac{1}{\log q}(\frac{1}{n}+\ldots+\frac{1}{n-j+1})]\ .$$

For j such that $1+\frac{1}{\log q}(\frac{1}{n}+\ldots+\frac{1}{n-j+1}) \leq 0$, we have

$\frac{\partial}{\partial\Delta} E(Y_j) \leq 0$. For j such that $1+\frac{1}{\log q}(\frac{1}{n}+\ldots+\frac{1}{n-j+1}) \geq 0$

we have $\frac{\partial}{\partial\Delta}E(Y_j) \geq 0$. Thus $E(Y_j)$ is maximized in the first

case at $\Delta = 0$ and at $\Delta = \xi_p$ in the second case. When $\Delta = 0$,

$E(Y_j) = \frac{\xi_p}{-\log q}\ (\frac{1}{n}+\ldots+\frac{1}{n-j+1})$. When $\Delta = \xi_p$, $E(Y_j) = \xi_p$.

To show (4.2). Let $\bar{G}(x) = \begin{cases} e^{x\log\frac{q}{\xi_p}} & \text{for}\ 0 \leq x < \xi_p \\ 0 & \text{for}\ \xi_p \leq x < \infty\ . \end{cases}$

Then $\bar{G}(x) \leq \bar{F}(x)$ for all $x \geq 0$ and G has p^{th} percentile ξ_p . Thus $E(Y_j) \leq E(X_j)$ where Y_j is the jth order statistic from G.. But

$$E(Y_j) = \int_0^\infty \sum_{i=0}^{j-1} \binom{n}{i} [G(x)]^i [\bar{G}(x)]^{n-i} dx .$$

Using the above definition of G , we obtain (4.2).

5. **Further results on order statistics and spacings.** In this section we present some results of theoretical interest concerning order statistics and their spacings from PF_2 and IFR (DFR) distributions. Many of the results hold without the restriction $F(0^-) = 0$.

First we present some total positivity properties of the order statistics. A function $K(x, y)$ of two real variables is said to be totally positive of order $r (TP_r)$ if for all $1 \leq m \leq r$; $x_1 \leq x_2 \leq \cdots \leq x_m$ and $y_1 \leq y_2 \leq \cdots \leq y_m$ we have the determinant inequalities

$$\left| K(x_i, y_j) \right|_{i, j=1}^{m} \geq 0 .$$

The following lemma is of use later on, as well as of interest in its own right.

Lemma 5.1. Let F be a distribution having density f with $f(x)$ not necessarily 0 for $x < 0$. Let $f_i(x)$ be the density of the ith order statistic in a sample of size n . Then $f_i(x)$ is TP_∞ in i, x where $i = 1, 2, \ldots, n$ and $-\infty < x < \infty$.

Proof.

(5.1) $$f_i(x) = \frac{n!}{(i-1)!(n-i)!} F^{i-1}(x) \bar{F}^{n-i}(x) f(x) .$$

Since $\left[\dfrac{F(x)}{\overline{F}(x)}\right]^{i-1}$ is TP_∞ in i and x, when $i = 1, 2, \ldots, n$

and $-\infty < x < \infty$, the conclusion follows. $\|$

We may obtain a similar result concerning the right hand tail of the distribution of an order statistic.

Lemma 5.2. Let F be any distribution with $F(x)$ not necessarily 0 for $x < 0$, F_i the corresponding distribution of the ith order statistic. Then $\overline{F}_i(x)$ is TP_∞ in i, x, where $i = 1, 2, \ldots, n$ and $-\infty < x < \infty$.

Proof.

$$(5.2) \qquad \overline{F}_i(x) = \sum_{j=0}^{i-1} \binom{n}{j} F^j(x)\, \overline{F}^{n-j}(x) .$$

Now $\left[\dfrac{F(x)}{\overline{F}(x)}\right]^j$ is TP_∞ in x, j. Therefore

$$\overline{F}_i(x) = \sum_{j=0}^{\infty} \binom{n}{j}\, \overline{F}^n(x) \left\{\dfrac{F(x)}{\overline{F}(x)}\right\}^j H(i-1-j) \quad \text{is also}$$

TP_∞ in i and x, where $H(k) = 1$ for $k \geq 0$, 0 otherwise. $\|$

In Barlow, Marshall, Proschan (1963), it is shown that the order statistics from an IFR distribution themselves have an IFR distribution. The next lemma shows a similar preservation of the PF_2 property.

Lemma 5.3. Suppose the underlying density f is PF_2, with $f(x)$ not necessarily 0 for $x < 0$. Then the density f_i of the ith order statistic is also PF_2 for fixed $i = 1, 2, \ldots, n$.

__Proof.__ It is easy to verify that when f is PF_2 , so is F and $\bar{F}$. Thus $\log f$, $\log F$, and $\log \bar{F}$ are concave. It follows from (5.1) that $\log f_i$ is concave, or equivalently, f_i is PF_2 for fixed $i = 1, 2, \ldots, n$. $\|$

Next we obtain some comparisons between the order statistics of an IFR (DFR) distribution and the corresponding order statistics of an exponential distribution.

__Theorem 5.4.__ Let $X_1 \le X_2 \le \ldots \le X_n$, $n \ge 2$, be order statistics from F , an IFR (DFR) distribution with mean θ , but $\not\equiv 1 - e^{-\frac{t}{\theta}}$. Let $Y_1 \le Y_2 \le \ldots \le Y_n$ be order statistics from $G(t) = 1 - e^{-\frac{t}{\theta'}}$. Then

(a) $EX_j - EY_j$ has at most one change of sign as j goes from 1 to n . Moreover if one change of sign does occur, then $EX_j - EY_j$ goes from positive (negative) to negative (positive) values.

(b) If $\theta = \theta'$, then one change of sign does occur.

__Proof.__

(a) Assume F is a continuous IFR distribution with mean θ and G is exponential with mean θ' . We have seen in Section 1 that if Y has distribution G , then $X = \phi(\frac{Y}{\theta'})$ has distribution F , where $\phi^{-1}(x) = -\log \bar{F}(x)$, a convex increasing function which is 0 for $x = 0$. Thus

$$EX_i - EY_i = \int_0^\infty \left\{ \phi\left(\frac{y}{\theta'}\right) - y \right\} g_i(y) dy ,$$

where g_i is the density of the ith order statistic from the exponential distribution G. By Lemma 5.1, $g_i(y)$ is TP_∞ in i and v. Also $\phi(\frac{y}{\theta}) - y$ changes sign at most once, and if once, from positive to negative values. Py the variation diminishing property of totally positive functions (Karlin, 1964, p. 34), $EX_i - EY_i$ also changes sign at most once, and from positive to negative values, if at all.

(b) If $\theta = \theta'$, then $\sum\limits_{i=1}^{n} EX_i = \theta = \theta' = \sum\limits_{i=1}^{n} EY_i$. Hence $EX_i - EY_i$

must change sign at least once or be identically 0 for $i = 1, 2, \ldots, n$. Now since $F \not\equiv G$, F cannot agree with G on an interval. Hence by Corollary 4.10, Chapter 2 of Barlow and Proschan (1964), $EX_1 > EY_1$ and $EX_n < EY_n$. Thus $EX_i - EY_i$ is not identically 0 for $i = 1, 2, \ldots, n$. Hence $EX_i - EY_i$ changes sign exactly once.

If F is IFR but not continuous, we may obtain the same result by using continuous IFR approximations.

Finally, a similar argument holds if F is DFR. $\|$

Actually, under the same hypothesis we may prove a stronger version of (a) in which $EX_i - EY_i$ is replaced by $EX_i^\alpha - EY_i^\alpha$, $\alpha > 0$. If instead of assuming $EX = EY$ in (b), we assume $EX^\alpha = EY^\alpha$, then we may show that one change of sign of $EX_i^\alpha - EY_i^\alpha$ does occur. We omit the details.

We may obtain further consequences of Theorem 5.4 using the notion of majorization. A vector $\underline{a} = (a_1, a_2, \ldots, a_n)$ majorizes a vector $\underline{b} = (b_1, b_2, \ldots, b_n)$ (written $\underline{a} > \underline{b}$) if

$$a_1 \geq a_2 \geq \ldots \geq a_n, \quad b_1 \geq b_2 \geq \ldots \geq b_n, \quad \sum_{i=1}^{k} a_i > \sum_{i=1}^{k} b_i,$$

$k = 1, 2, \ldots, n-1,$ and $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i$. See Hardy, Littlewood,

Pólya (1952), Chapter II.

<u>Theorem 5.5.</u> Let $X_1 \leq \ldots \leq X_n$ be order statistics from F ,
an IFR (DFR) distribution with mean θ , $Y_1 \leq \ldots \leq Y_n$ be
order statistics from $G(t) = 1 - e^{-\frac{t}{\theta}}$. Then $(EY_n, EY_{n-1}, \ldots, EY_1)$
$>(<)(EX_n, EX_{n-1}, \ldots, EX_1)$.

<u>Proof.</u> Let F be IFR . From Theorem 5.4 we know

$EY_{n-i+1} - EX_{n-i+1}$ has one change of sign, from plus to minus as

i goes from 1 to n . We also know $\sum_{i=1}^{n} EY_{n-i+1} = n\theta = \sum_{i=1}^{n} EX_{n-i+1}.$

Thus $\sum_{i=1}^{j} EY_{n-i+1} \geq \sum_{i=1}^{j} EX_{n-i+1},$ $j = 1, 2, \ldots, n$. Finally, EY_{n-i+1}

and EX_{n-i+1} are decreasing in i . Thus the conclusion follows.

A similar argument holds if F is DFR . $\|$

Using Karamata's Theorem we obtain Theorem 5.6 below.

Karamata's Theorem states that if ψ is continuous and convex and

$\underline{a} > \underline{b}$, then

(5.3) $$\sum_{i=1}^{n} \psi(a_i) \geq \sum_{i=1}^{n} \psi(b_i) .$$

See Hardy, Littlewood, Pólya (1952), p.89.

<u>Theorem 5.6.</u> Let ψ be continuous and convex, $X_1, \ldots, X_n,$
$Y_1, \ldots, Y_n$ as in Theorem 5.5. Then

$$\sum_{i=1}^{n} \psi(EY_i) \geq \sum_{i=1}^{n} \psi(EX_i) .$$

Proof. Let F be IFR . By Theorem 5.5, $(FY_n, \ldots, FY_1)$ > $(FX_n, \ldots, FX_1)$. Hence by Karamata's Theorem, the conclusion follows. A similar argument holds if F is DFR . |

Using Theorem 5.5 we obtain

Theorem 5.7. Let $c_1 \leq c_2 \leq \cdots \leq c_n$, $X_1, \ldots, X_n, Y_1, \ldots Y_n$ as in Theorem 5.5. Then $\sum_{i=1}^{n} c_i FY_i \geq \sum_{i=1}^{n} c_i FX_i$.

Proof. Let F be IFR . Defining $d_i = FY_{n-i+1} - FX_{n-i+1}$, write

$$\sum_{i=1}^{n} c_{n-i+1}(FY_{n-i+1} - FX_{n-i+1}) = (c_n - c_{n-1})d_1 + (c_{n-1} - c_{n-2})(d_1 + d_2)$$

$$+ (c_{n-2} - c_{n-3})(d_1 + d_2 + d_3) + \ldots + (c_2 - c_1)(d_1 + \ldots + d_{n-1})$$

$$+ c_1(d_1 + \ldots + d_n) .$$

Since $c_i - c_{i+1} \geq 0$, $i = 1, \ldots, n-1$, $d_1 + \ldots + d_j \geq 0$, $j = 1, \ldots, n-1$, and $d_1 + \ldots + d_n = 0$, we conclude that

$$\sum_{i=1}^{n} c_{n-i+1}(FY_{n-i+1} - FX_{n-i+1}) \geq 0 .$$

A similar argument holds when F is DFR. ||

Finally we summarize some results concerning the covariance of order statistics obtained by Tukey (1958). He shows that if F is IFR , then for $h \leq j < k$, $\text{cov}(X_k, X_h) < \text{cov}(X_j, X_h)$, where X_j is the jth order statistic from F . He further shows that if F satisfies both

(a) $\log \bar{F}$ is concave (i.e., F is IFR), and

(b) $\log F$ is concave,

then

(1) the covariance of any two order statistics is less than the variance of either, and

(2) the covariance between order statistics X_j, X_k is monotone in j and k separately, decreasing as j and k separate from one another.

Note that if f is PF_2 , then F satisfies both (a) and (b) above.

Next we derive properties of the spacings $X_1, X_2 - X_1, \ldots,$ $X_n - X_{n-1}$ from PF_2 and IFR (DFR) distributions similar to those of the order statistics $X_1 \leq X_2 \leq \ldots \leq X_n$ obtained above. We first consider total positivity properties.

<u>Theorem 5.8.</u> Let f be PF_2 with $f(x)$ not necessarily 0 for $x < 0$. Then h_i , the density of $X_i - X_{i-1}$, is PF_2 for fixed $i = 2, 3, \ldots, n.$ If we assume further that $f(x) = 0$ for $x < 0$,, then h_1 is PF_2 , where h_1 is the density of X_1 .

<u>Proof.</u>

$$(5.4) \qquad h_i(x) = \frac{n!}{(i-2)!(n-i)!} \int F^{i-2}(u) f(u) f(u+x) \bar{F}^{n-i}(u+x) \, du$$

for $i = 2, 3, \ldots, n$.

Since f is PF_2 , so is $r(u) = F^{i-2}(-u) f(-u)$, $s(u) = f(u) \bar{F}^{n-i+1}(u)$. Hence so is

$$h_i(x) = \frac{n!}{(i-1)!(n-i+1)!} \int r(-u) s(u+x) \, du$$

for fixed $i = 2, 3, \ldots, n$.

Assuming $f(x) = 0$ for $x < 0$, we see that h_1 is PF_2, from

$$(5.5) \qquad h_1(x) = n f(x) \bar{F}^{n-1}(x) \cdot \|$$

<u>Theorem 5.9.</u> If F is DFR , then H_1 is DFR for fixed $i = 1, 2, \ldots, n$.

<u>Proof.</u> Since DFR is preserved under convex combinations (Barlow, Marshall, Proschan, 1963, p. 381) we see from the representation

(5.6) $$\Pi_i(x) = \frac{n!}{(i-1)!(n-i)!} \int F^{i-1}(u) \, f(u) \, \bar{F}_u^{n-1}(x) \, du,$$

where $F_u(x) = F(u + x)$, DFR in x for fixed u , that H_i is DFR for fixed $i = 1, 2, \ldots, n.$ $\|$

<u>Theorem 5.10.</u> Let F be DFR with $F(x) < 1$ for all $x > 0$. Then $\Pi_i(x)$ is TP_2 in i, x where $i = 2, 3, \ldots, n$ and $x \geq 0$.

<u>Proof.</u> $F^{i-2}(u) \, f(u)$ is TP_2 in $i = 2, 3, \ldots, n$ and $u \geq 0$. $\bar{F}^{n-i+1}(u+x)$ is TP_2 in i, u, in i, x , and in $u, x \, (u \geq 0, x \geq 0)$. Thus by a theorem[*] in the book by Karlin (in process)

(5.6) $$\Pi_i(x) = \frac{n!}{(i-1)!(n-i)!} \int F^{i-2}(u) \, f(u) \, \bar{F}^{n-i+1}(u+x) \, du$$

is TP_2 in i, x , where $i = 2, 3, \ldots, n$ and $x \geq 0$.

---

[*] <u>Theorem</u> (Karlin) let λ, x, ξ traverse linear sets Λ, X, Ξ respectively. Suppose $h(\lambda, x) = \int f(\lambda, x, \xi) \, g(\lambda, \xi) \, d\mu(\xi)$ is well defined on $\Lambda \times X$, where μ is a σ-finite measure, and

(i) $f(\lambda, x, \xi) > 0$ for all λ in Λ , x in X , and ξ in Ξ , and f is TP_2 for each pair of variables when the third variable is held fixed.

(ii) $g(\lambda, \xi)$ is TP_2 .

Then h is TP_2 .

Next we present some majorization properties of the nor-
malized spacings $(n - i + 1)(X_i - X_{i-1})$, $i = 1, 2, \ldots, n$, similar to
these developed above for the order statistics.

Theorem 5.11. Let $X_1 \leq X_2 \leq \ldots \leq X_n$ be the order statistics
from F , an IFR distribution with mean μ , $Y_1 \leq Y_2 \leq \ldots \leq Y_n$
the order statistics from $G(t) = 1 - e^{-\frac{t}{\mu}}$. Then

$$EnX_1, E(n - 1)(X_2 - X_1), \ldots, E(X_n - X_{n-1})$$
$$\geq (\leq) \; EnY_1, E(n - 1)(Y_2 - Y_1), \ldots, E(Y_n - Y_{n-1}) \; .$$

Proof. Let F be IFR . By Theorem 2.1, $F(n - i + 1)(X_i - X_{i-1})$
is decreasing in i . It is also easy to verify that EnY_1
$= E(n - 1)(Y_2 - Y_1) = \ldots = E(Y_n - Y_{n-1})$. Since

$$\sum_{i=1}^{n} E(n - i + 1)(X_i - X_{i-1}) = \mu = \sum_{i=1}^{n} E(n - i + 1)(Y_i - Y_{i-1}) \; ,$$

it follows that

$$\sum_{i=1}^{j} E(n - i + 1)(X_i - X_{i-1}) \geq \sum_{i=1}^{j} E(n - i + 1)(Y_i - Y_{i-1})$$

for $j = 1, 2, \ldots, n-1$. Thus the conclusion follows.

A similar argument holds if F is DFR.

For normalized spacings, the analogue of Theorem 5.7 is

Theorem 5.12. Let $c_1 \geq \ldots \geq c_n$, $X_1, X_2, \ldots, X_n$, $Y_1, Y_2, \ldots, Y_n$
as in Theorem 5.11. Then

$$\sum_{i=1}^{n} c_i E(n - i + 1)(X_i - X_{i-1}) \geq_{(\leq)} \sum_{i=1}^{n} c_i E(n - i + 1)(Y_i - Y_{i-1}) \; .$$

Proof. The proof parallels that of Theorem 5.7.

　　We immediately obtain:

Corollary 5.13. Let $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ be as in Theorem 5.11. Then for $1 \le r \le n$

$$\sum_{i=1}^{r} E(n-i+1)(X_i - X_{i-1}) \ge \sum_{i=1}^{r} E(n-i+1)(Y_i - Y_{i-1}) \ .$$

Proof. Choose $c_1 = c_2 = \ldots = c_r = 1, c_{r+1} = c_{r+2} = \ldots = c_n = 0$, so that $c_1 \ge c_2 \ge \ldots \ge c_n$. The result follows from Theorem 5.11.

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This paper studies the performance of the standard life test procedures developed for the exponential distribution when the underlying distribution actually has an increasing (decreasing) failure rate. The standard exponential life test procedures are shown to yield estimates of the unknown mean biased on the high side in the increasing failure rate case. The practical implications of these results are that the consumer tends to accept unduly often poor quality lots, when the producer has used an exponential life test procedure and the underlying failure distribution actually had increasing failure rate.

In addition a number of theoretical results are obtained concerning order statistics and their spacings from distributions with increasing failure rate. These results are useful in the study of the life test procedures and also are of independent interest.

14. KEY WORDS	LINK A		LINK B		LINK C	
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